Math 630-102
Homework #10
Due date: April 12, 2007

Group work on h/w assignments is not allowed. No credit is given for results without a solution or an explanation. Late homework is not accepted.

Section 5.1

Problem I. Calculate the eigenvalues and the eigenvectors of the following matrices, and compare the two sets of results. Verify that the trace of each matrix (sum of diagonal elements) equals the sum of its eigenvalues, and the determinant equals the product of the eigenvalues.

$$A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}, \quad A + 2\,I = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

Problem II. Find the eigenvalues and the eigenvectors of the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. (note that this is a triangular matrix, which greatly simplifies the characteristic equation). Recall that an eigenvector of a given eigenvalue λ is the nullspace $N(A - \lambda\,I)$

Section 5.2

Problem III. The following matrices have only one distinct eigenvalue. However, one of these matrices has two linearly independent eigenvectors, while the other has only one. Explain this fact based on the rank of $(A - \lambda\,I)$. Are these matrices diagonalizable?

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Problem IV. For a matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, calculate the following:

a) Eigenvectors and eigenvalues
b) Matrix S composed of the eigenvectors of A, and its inverse, S^{-1}. Verify using matrix multiplication that $A = S\,\Lambda\,S^{-1}$

c) Use the diagonalization $A = S \Lambda S^{-1}$ to show that $A^k = \dfrac{1}{2}\begin{bmatrix} 3^k + 1 & 3^k - 1 \\ 3^k - 1 & 3^k + 1 \end{bmatrix}$

Problem V. Given the diagonalization $A = S \Lambda S^{-1}$, write the expression for the diagonalization of matrix A^{-1}. What are the eigenvalues of A^{-1}, given the eigenvalues λ_j of the original matrix A?

Section 5.3

Problem VI. Consider a biennial plant which produces on average 3 seeds per each one-year-old plant ("young"), and 2 seeds per each two-year-old plant ("adult"). Further, assume that the survival (germination) probability for each seed is ¼, and that each one-year-old plant has only a 1/2 chance of surviving to year two.

a) Show that the plant number evolves according to the equation $u_k = A\, u_{k-1}$, where
$A=\begin{bmatrix} 3/4 & 1/2 \\ 1/2 & 0 \end{bmatrix}$, and u_k is the vector containing the number of "young" and "adult" plants at year k (one sentence is sufficient). Thus, $u_k = A^k\, u_o$, where u_o is the initial number of plants.

b) Find the eigenvalues and the eigenvectors of the system matrix (no irrational numbers involved!). Determine the diagonalization $A = S \Lambda S^{-1}$ (verify your answer by matrix multiplication)

c) Based on the eigenvalues alone, can you say whether the population size will grow?

d) Starting with 20 seeds at time $k=0$, how many one- and two-year-old plants will there be after $k=40$ years? Start by expressing this initial state u_o as a linear combination of the eigenvectors, using the coefficients $c = S^{-1}u_o$ (consult the summary on last page if in doubt).

Summary:

- Eigenvalues λ and eigenvectors x satisfy the equation $\mathbf{A}\,x = \lambda\,x$

- Eigenvectors are really "eigenspaces" ("eigendirections") that are not affected by the action of A (apart from uniform stretching by factor λ). Any multiple of an eigenvector is also an eigenvector.

- Eigenvalues are found by solving the *characteristic equation* $\mathbf{det(A - \lambda\,I) = 0}$

- The characteristic equation for an $n{\times}n$ matrix is an n-th degree polynomial, and therefore always has n roots. However, the roots (eigenvalues) are not necessarily real, and more importantly, not necessarily distinct

- The following two results are particularly helpful in the 2×2 case:

    $\mathbf{det\ A} = \lambda_1 \cdot \lambda_2 \cdot\ ...\cdot \lambda_n$ (product of eigenvalues)
    $\mathbf{trace\ A} = \lambda_1 + \lambda_2 +\ ...+ \lambda_n$ (sum of eigenvalues)

- Once the eigenvalues are found, the eigenvectors can be determined by solving

    $\mathbf{A}\,x = \lambda\,x\quad$ or an equivalent equation:
    $\mathbf{(A - \lambda\,I)}\,x = \mathbf{0}$

    Therefore, an eigenvector x with an eigenvalue λ equals the nullspace $N(A - \lambda\,I)$
    Recall that in determining the null-space, we set the free variable (one of the components of the eigenvector) to 1.

- If a matrix has n distinct linearly independent eigenvectors, it can be **diagonalized**

    $\mathbf{A = S\,\Lambda\,S^{-1}}$

    Where the columns of matrix S are the eigenvectors, and Λ is the diagonal matrix composed of the eigenvalues. Note that the matrix S is not unique since each of the eigenvectors can be multiplied by any constant.

- If A can be diagonalized, we can find an arbitrary power of A according to

    $\mathbf{A^k = S\,\Lambda^k\,S^{-1}}$

    Where Λ^k is a matrix with k-th powers of eigenvalues along the diagonal

Difference equations:

- **A** difference equation is an equation of form $u_{k+1} = A u_k$, where A is a matrix describing the system, and integer k can be viewed as a discrete time step.

- The solution to the difference equation is $u_k = A^k u_o$, where u_o is the initial state of the system.

- If matrix A can be diagonalized, the solution can be written as

$$u_k = S \Lambda^k S^{-1} u_o$$

or, equivalently,

$$u_k = S \Lambda^k c = c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2 + \ldots + c_n \lambda_n^k x_n$$

where $c = S^{-1} u_o$ is the vector of coefficients in the expansion of u_o as a linear combination of eigenvectors:

$$u_o = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n = S c$$